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TORELLI THEOREM FOR SURFACES WITH $p_g = c_1^2 = 1$ AND K AMPLE AND WITH CERTAIN TYPE OF AUTOMORPHISM

Sampei Usui

0. Introduction

The moduli space of isomorphism classes of surfaces with $p_g = c_1^2 = 1$ is studied by Catanese in [2]. Every such surface with the ample canonical divisor can be represented as a smooth weighted complete intersection of type $(6, 6)$ in $\mathbf{P} = \mathbf{P}(1, 2, 2, 3, 3)$ parametrized by a Zariski open set $U \subset \mathbf{A}^{26}$ (cf. (1.3)). This leads to a universal family

$$\pi' : \mathscr{X}' \to U.$$

There is an 8-dimensional subroup G of $\mathrm{Aut}(\mathbf{P})$ (cf. (1.5) and (1.6)) acting on U with finite isotropy groups and

$$M = U/G = \begin{matrix} \text{the moduli space of canonical surfaces} \\ \text{with } p_g = c_1^2 = 1. \end{matrix}$$

In particular, $\dim_{\mathbf{C}} M = 18$.

The period domain D, which parametrizes polarized Hodge structures on the second primitive cohomology groups of the surfaces in question, is isomorphic to

$$\{[a] \in \mathbf{P}(L \otimes \mathbf{C}) \mid (a, a) = 0, (a, \bar{a}) > 0\}$$

where L is a free $\mathbf{Z}$-module of rank 20 equipped with a symmetric bilinear form $(\ , \ )$ of signature $(2, 18)$. The group $\Gamma = \mathrm{Aut}(L)$ acts properly discontinuously on D.

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Set

$$\tilde{U} = \{(u, \alpha) \mid u \in U, \alpha \in \mathrm{Isom}(P^2(X_u, \mathbf{Z}), L)\} \quad \text{and} \quad \tilde{\mathscr{X}}' = \mathscr{X}' \times_U \tilde{U}.$$

Then we have the universal family

$$\tilde{\pi} : \tilde{\mathscr{X}} = \tilde{\mathscr{X}}'/G \to \tilde{M} = \tilde{U}/G \tag{0.1}$$

of marked canonical surfaces with $p_g = c_1^2 = 1$ (cf. Proposition (2.24) in [11]). $\tilde{M}$ and $\tilde{\mathscr{X}}$ are complex manifolds and this family serves as a universal family of the deformations of the surfaces in question. This gives a period map

$$\Phi : \tilde{M} \to D.$$

Catanese has shown in [2] (cf. also [12]) that Φ has non-empty ramification locus $\tilde{\Delta} \subset \tilde{M}$. Thus the local Torelli fails at $\tilde{m} \in \tilde{\Delta}$. The problem then is to study how badly it can fail. First of all observe that

$$\dim \operatorname{Ker} d\Phi(\tilde{m}) \leq 2.$$

This directly follows from the exact sequence

$$0 \longrightarrow H^0(C_{\tilde{m}}, \Omega^1_{X_{\tilde{m}}} \otimes \mathcal{O}_{C_{\tilde{m}}}) \longrightarrow H^1(X_{\tilde{m}}, T_{X_{\tilde{m}}}) \xrightarrow{d\Phi(m)} H^1(X_{\tilde{m}}, \Omega^1_{X_{\tilde{m}}})$$

together with the fact that $h^0(C_{\tilde{m}}, \Omega^1_{X_{\tilde{m}}} \otimes \mathcal{O}_{C_{\tilde{m}}}) \leq h^0(C_{\tilde{m}}, \Omega^1_{C_{\tilde{m}}}) = 2$, where $C_{\tilde{m}}$ is the canonical curve of $X_{\tilde{m}}$. This means that the fibre of Φ through $\tilde{m} \in \tilde{M}$ has at most dimension 2. Todorov ([9]) and the author ([10]) have shown that this indeed happens for certain surfaces $X_{\tilde{m}}$ which are double coverings of K3 surfaces.

We have classified in [11] the automorphisms of the surfaces in question and shown, in particular, that any automorphism of prime order of the surfaces in question is conjugate to one of σ_1, σ_3, σ_8, σ_{11}, σ_{15}, $\sigma_{0'} \in \mathrm{Aut}(\mathbf{P})$, which are defined respectively by

$$\begin{aligned}
\sigma_1(x_0, y_1, y_2, z_3, z_4) &= (x_0, y_1, y_2, z_3, -z_4)\\
\sigma_3(x_0, y_1, y_2, z_3, z_4) &= (x_0, y_1, y_2, -z_3, -z_4)\\
\sigma_8(x_0, y_1, y_2, z_3, z_4) &= (x_0, \omega y_1, y_2, z_3, z_4)\\
\sigma_{11}(x_0, y_1, y_2, z_3, z_4) &= (x_0, \omega y_1, \omega y_2, x_3, z_4)\\
\sigma_{15}(x_0, y_1, y_2, z_3, z_4) &= (x_0, \omega y_1, \omega^2 y_2, z_3, z_4)\\
\sigma_{0'}(x_0, y_1, y_2, z_3, z_4) &= (x_0, y_1, -y_2, z_4, z_3)
\end{aligned}$$

where x_0, y_1, y_2, z_3 and z_4 are weighted homogeneous coordinates of $\mathbf{P}(1,2,2,3,3)$ and $\omega = \exp(2\pi i/3)$. By using this classification, we have shown:

$$\begin{array}{lcl} \Phi \text{ has the 2-dimensional} & & \exists\sigma \in \mathrm{Aut}(X_{\tilde{m}}) \text{ which is} \\ \text{fibre through } \tilde{m} \in \tilde{M} & \Leftrightarrow & \text{conjugate to } \sigma_3, \end{array}$$

$$\begin{array}{lcl} \Phi \text{ has the positive dimensional} & & \exists\sigma \in \mathrm{Aut}(X_{\tilde{m}}) \text{ which is} \\ \text{fibre through } \tilde{m} \in \tilde{M} & \Leftarrow & \text{conjugate to } \sigma_1 \text{ or } \sigma_8 \end{array}$$

(see, for detail, [10] and [11]).

In this paper, we investigate those canonical surfaces with $p_g = c_1^2 = 1$ which have automorphisms conjugate to σ_{15}. Let M_{15} be the set of isomorphism classes of these surfaces. After our classification in [11], we have:

M_{15} = the set of isomorphism classes of canonical surfaces with $p_g = c_1^2 = 1$ and with an automorphism of order 3 acting trivially on the holomorphic 2-forms.

Set $\sigma = \sigma_{15}$ and let us consider smooth weighted complete intersections of type $(6,6)$ in $\mathbf{P} = \mathbf{P}(1,2,2,3,3)$ with defining equations

$$\text{(0.2)} \qquad \begin{cases} f = z_3^2 + f_0 z_4 x_0^3 + f_{111} y_1^3 + f_{222} y_2^3 + f_{012} x_0^2 y_1 y_2 + f_{000} x_0^6, \\ g = z_4^2 + g_0 z_3 x_0^3 + g_{111} y_1^3 + g_{222} y_2^3 + g_{012} x_0^2 y_1 y_2 + g_{000} x_0^6. \end{cases}$$

These surfaces are stable under the action of σ. Denote by

$$\text{(0.3)} \qquad \pi'_{15} : \mathscr{X}'_{15} \to U_{15}$$

the smooth family of weighted complete intersections of type $(6,6)$ in $\mathbf{P}(1,2,2,3,3)$ with equations (0.2) parametrized by their 10 coefficients

$$(f_0, f_{111}, f_{222}, f_{012}, f_{000}, g_0, g_{111}, g_{222}, g_{012}, g_{000}) \in U_{15} \subset \mathbf{A}^{10}.$$

The automorphism $\sigma \in \mathrm{Aut}(\mathbf{P})$ has the induced action on the family (0.3) which is trivial on the parameter space U_{15}. We abuse the notation σ for indicating the induced automorphism of each fibre $X_u = \pi'^{-1}_{15}(u)$ $(u \in U_{15})$.

There exists a 4-dimensional subgroup $H \subset G \subset \mathrm{Aut}(\mathbf{P})$ (cf. (1.12)) and our Proposition (1.14) asserts that

$$U_{15}/H \xrightarrow{\sim} M_{15} \quad \text{(and hence } \dim M_{15} = 6)$$

sending $u \in U_{15}$ to the isomorphism class containing X_u, and that, for any $X \in M_{15}$ and for any automorphism α of X of order 3 acting trivially on $H^0(X, K_X)$, there exists a point $u \in U_{15}$ and an isomorphism $\tau : X_u \xrightarrow{\sim} X$ such that $\alpha = \tau\sigma\tau^{-1}$.

Let $u_k \in U_{15}$ and set $X_k = X_{u_k}$ $(k = 1, 2)$. Take a basis ω_{X_k} of $H^0(X_k, K_{X_k})$. Set

$$H_2(X_k, \mathbf{Z})^\sigma = \mathrm{Ker}\{1 - \sigma : H_2(X_k, \mathbf{Z}) \to H_2(X_k, \mathbf{Z})\}.$$

Now our main theorem in the present paper is stated as follows:

THEOREM (3.4): *Let $u_k \in U_{15}$ $(k = 1, 2)$. Suppose that there exists a path $\tilde{\tau}$ in U_{15} joining u_1 and u_2 which induces an isometry*

$$\tau_* : H_2(X_1, \mathbf{Z})^\sigma \to H_2(X_2, \mathbf{Z})^\sigma$$

preserving the periods of integrals of the holomorphic 2-forms ω_{X_k} on X_k, i.e.

$$\int_{\tau_*\gamma} \omega_{X_2} = (constant) \int_\gamma \omega_{X_1} \quad \textit{for all } \gamma \in H_2(X_1, \mathbf{Z}),$$

where (constant) is independent of γ.

Then, there exists an isomorphism

$$\tau : X_1 \to X_2$$

inducing the given isometry τ_* and such τ is uniquely determined up to composition with an element of the group $\langle\sigma\rangle$ generated by σ. We have also $\tau\sigma\tau^{-1} = \sigma$ or σ^2.

Roughly speaking, Theorem (3.4) is proved by applying the Strong Torelli Theorem for algebraic K3 surfaces (cf. [8], [1] and [7]) to the K3 surfaces obtained as the desingularizations of $X_u/\langle\sigma\rangle$ $(u \in U_{15})$.

Our present results can be rephrased in the language of period map as follows. Fix a base point $u_0 \in U_{15}$ and identify $P^2(X_{u_0}, \mathbf{Z}) = L$. Set

$$\tilde{U}_{15} = \left\{(u, \tau_*) \,\middle|\, \begin{array}{c} u \in U_{15},\ \tau_* \in \mathrm{Isom}(P^2(X_u, \mathbf{Z}), L) \text{ coming from a path} \\ \tilde{\tau} \text{ joining } u \text{ and } u_0 \text{ in } U_{15} \end{array}\right\}$$

and

$$\tilde{\mathscr{X}}'_{15} = \mathscr{X}'_{15} \underset{U_{15}}{\times} \tilde{U}_{15}.$$

Note that the fibre of $\tilde{U}_{15} \to U_{15}$ is the geometric monodromy group $\Gamma_{U_{15}} = \mathrm{Im}\{\pi_1(U_{15}) \to \mathrm{Aut}(L)\}$. Then we have, as in a similar way as (0.1), the universal family

$$\tilde{\pi}_{15} : \tilde{\mathscr{X}}_{15} = \tilde{\mathscr{X}}'_{15}/H \to \tilde{M}_{15} = \tilde{U}_{15}/H$$

and the period map

$$\Phi_{15} : \tilde{M}_{15} \to D.$$

Φ_{15} induces a set-theoretic map

$$\bar{\Phi}_{15} : M_{15} \to D/\Gamma_{U_{15}}.$$

Our Proposition (1.17) and Theorem (3.4) assert that Φ_{15} is unramified and $\bar{\Phi}_{15}$ is injective.

The following are unknown at present:

(0.4) Whether Φ_{15} is an immersion.
(0.5) The description of the difference of $\Gamma_{U_{15}}$ and $\Gamma = \mathrm{Aut}(L)$.
(0.6) The determination of the image of Φ_{15}.
(0.7) The study of the surfaces with automorphisms conjugate to σ_{11} or to $\sigma_{0'}$.
(0.8) The determination of all the points of $\tilde{M}$ through which Φ has 1-dimensional fibres.

Every variety in this paper is a variety over the field $\mathbf{C}$ of complex numbers.

1. Surfaces with $p_g = c_1^2 = 1$

1.1. F. Catanese showed in [2] that the canonical models of the surfaces with $p_g = c_1^2 = 1$ are represented as weighted complete intersections of type (6, 6) in $\mathbf{P} = \mathbf{P}(1, 2, 2, 3, 3)$. If we assume furthermore that the canonical invertible sheaf K_X of the surface X in question is ample, the canonical model of X is smooth and hence we can identify X with its canonical model.

Let $R = \mathbf{C}[x_0, y_1, y_2, z_3, z_4]$ be the weighted polynomial ring with $\deg x_0 = 1$, $\deg y_1 = \deg y_2 = 2$ and $\deg z_3 = \deg z_4 = 3$. Catanese also showed that the defining equations of the canonical models in question are partially normalized as follows (cf. [2]):

$$(1.1) \qquad \begin{cases} f = z_3^2 + f^{(1)} z_4 x_0 + f^{(3)}, \\ g = z_4^2 + g^{(1)} z_3 x_0 + g^{(3)}, \end{cases}$$

where $f^{(1)}$ and $g^{(1)}$ are linear and $f^{(3)}$ and $g^{(3)}$ are cubic forms in x_0^2, y_1 and y_2, i.e., by using the notation $y_0 = x_0^2$,

$$f^{(1)} = \sum_{0 \leqq i \leqq 2} f_i y_i, \quad f^{(3)} = \sum_{0 \leqq i \leqq j \leqq k \leqq 2} f_{ijk} y_i y_j y_k,$$
(1.2)
$$g^{(1)} = \sum_{0 \leqq i \leqq 2} g_i y_i, \quad g^{(3)} = \sum_{0 \leqq i \leqq j \leqq k \leqq 2} g_{ijk} y_i y_j y_k.$$

Varying these 26 coefficients f_i, f_{ijk}, g_i and g_{ijk}, we get a family of weighted complete intersections in $\mathbf{P} = \mathbf{P}(1, 2, 2, 3, 3)$. Set

(1.3)
$$U = \left\{ u \in \mathbf{A}^{26} \,\middle|\, \begin{array}{l} \textit{the corresponding surface is a} \\ \textit{smooth weighted complete intersections} \\ \textit{of type } (6,6) \textit{ in } \mathbf{P}(1,2,2,3,3) \end{array} \right\}$$

and let

(1.4)
$$\mathscr{X}' \to U$$

be the family of the surfaces in $\mathbf{P}(1, 2, 2, 3, 3)$. Note that U is a Zariski open subset of $\mathbf{A}^{26}$.

Let G be the group consisting of the non-degenerate matrices over $\mathbf{C}$ of the forms

(1.5)
$$\left(\begin{array}{c|cc|cc} d_0 & & & & \\ \hline d_{10} & d_{11} & d_{12} & & \\ d_{20} & d_{21} & d_{22} & & \\ \hline & & & d_3 & 0 \\ & & & 0 & d_4 \end{array}\right)$$

and

(1.6)
$$\left(\begin{array}{c|cc|cc} d_0 & & & & \\ \hline d_{10} & d_{11} & d_{12} & & \\ d_{20} & d_{21} & d_{22} & & \\ \hline & & & 0 & d_3 \\ & & & d_4 & 0 \end{array}\right)$$

acting on $\mathbf{P}(1, 2, 2, 3, 3)$ as

$$\begin{cases} x_0 \mapsto d_0 x_0 & \\ y_i \mapsto \sum_{0 \leqq j \leqq 2} d_{ij} y_j & (i = 1, 2) \\ z_i \mapsto d_i z_i & (i = 3, 4) \end{cases}$$

in case (1.5), and

$$\begin{cases} x_0 \mapsto d_0 x_0 & \\ y_i \mapsto \sum_{0 \leqq j \leqq 2} d_{ij} y_j & (i = 1, 2) \\ z_3 \mapsto d_3 z_4 & \\ z_4 \mapsto d_4 z_3 & \end{cases}$$

in case (1.6).

Since the canonical invertible sheaves of the surfaces X_u $(u \in U)$ are isomorphic to $\mathcal{O}_{X_u}(1)$ and their defining equations are partially normalized as (1.1), we can prove easily that every isomorphism between the surfaces X_u $(u \in U)$ is induced from some element in G (see, for detail, [2] or [11]). Hence we see, by [4], that

(1.7) $U/G =$ the coarse moduli scheme of complete, smooth surfaces with $p_g = c_1^2 = 1$ and K ample.

1.2. In [11], we classified the automorphisms of the surfaces X with $p_g = c_1^2 = 1$ and K_X ample, and determined the induced action on $H^2(X, \mathbf{C})$, on $H^{2,0}(X)$ and on $H^1(X, T_X)$.

Among these automorphisms we are mainly interested in the present paper in σ_{15} in Theorem (2.14) in [11]. We fix, throughout this paper, the notation

(1.8) $$\sigma = \sigma_{15} = (1, \omega, \omega^2, 1, 1) \in G$$

which means the diagonal matrix

$$\sigma = \boxed{\begin{matrix} 1 & & & & \\ & \omega & & 0 & \\ & & \omega^2 & & \\ & 0 & & 1 & \\ & & & & 1 \end{matrix}}, \text{ where } \omega = \exp(2\pi\sqrt{-1}/3).$$

Set

$$U_{15} = \{u \in U \mid \sigma u = u\} \tag{1.9}$$

and denote by

$$\pi'_{15} : \mathscr{X}'_{15} \to U_{15} \tag{1.10}$$

the family induced from (1.4) by $U_{15} \hookrightarrow U$. More explicitly, the defining equations of the surfaces $X_u = \pi'^{-1}_{15}(u)$ $(u \in U_{15})$ have the following forms:

$$\begin{cases} f = z_3^2 + f_0 z_4 x_0^3 + f_{111} y_1^3 + f_{222} y_2^3 + f_{012} x_0^2 y_1 y_2 + f_{000} x_0^6, \\ g = z_4^2 + g_0 z_3 x_0^3 + g_{111} y_1^3 + g_{222} y_2^3 + g_{012} x_0^2 y_1 y_2 + g_{000} x_0^6. \end{cases} \tag{1.11}$$

Define

$$H = \{\tau \in G \mid \tau(U_{15}) \cap U_{15} \neq \emptyset\}.$$

By an elementary calculation using (1.11), we can prove that H consists of the following four types of matrices:

$$\begin{pmatrix} d_0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 \\ 0 & 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 0 & d_4 \end{pmatrix} \qquad \begin{pmatrix} d_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_1 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & 0 & d_4 \end{pmatrix}$$

$$\begin{pmatrix} d_0 & 0 & 0 & 0 & 0 \\ 0 & d_1 & 0 & 0 & 0 \\ 0 & 0 & d_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & d_4 & 0 \end{pmatrix} \qquad \begin{pmatrix} d_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_1 & 0 & 0 \\ 0 & d_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & d_4 & 0 \end{pmatrix} \tag{1.12}$$

We can also prove, by using the forms (1.12), that H is the normalizer of $\langle\sigma\rangle$ in G, where $\langle\sigma\rangle$ is the subgroup of G generated by σ in (1.8).

Set

(1.13) M_{15} = the set of the isomorphism classes of the complete, smooth surfaces with $p_g = c_1^2 = 1$ and K ample and with an automorphism of order 3 acting trivially on the holomorphic 2-forms.

PROPOSITION (1.14): *We have a natural bijection* $U_{15}/H \simeq M_{15}$ *as sets and* U_{15}/H *is a 6-dimensional irreducible subvariety of the coarse moduli space* U/G *in* (1.7). *Moreover, for any surface* $X \in M_{15}$ *and for any automorphism* α *of* X *of order* 3 *acting trivially on* $H^0(X, K_X)$, *there exist a point* $u \in U_{15}$ *and an isomorphism* $\tau : X_u \xrightarrow{\sim} X$ *satisfying* $\alpha = \tau\sigma\tau^{-1}$.

PROOF: This is an immediate consequence of Theorem (2.14) in [11]. Note that "natural" in the statement of the proposition means that H-orbit of $u \in U_{15}$ corresponds to the isomorphism class containing X_u. Q.E.D.

1.3. Let $X = X_u$ for some $u \in U_{15}$ and let S be the parameter space of the Kuranishi family of the deformations of $X = X_{s_0}$ ($s_0 \in S$).

S is smooth and the Kuranishi family is universal (see, for detail, [11]). Hence, $\sigma \in \mathrm{Aut}(X)$ has the induced action on S via the identification $X = X_{s_0}$. Set

$$S^\sigma = \{s \in S \mid \sigma s = s\}. \tag{1.15}$$

Note that, since σ is of finite order, S^σ is a submanifold of S. Note also that S^σ is the parameter space of the universal family of the deformations of the pair (X, σ) of the surface X and $\sigma \in \mathrm{Aut}(X)$.

Let

$$\phi : S \to D \tag{1.16}$$

be the period map, using the Hodge decomposition of the second primitive cohomology group $P^2(X_s, \mathrm{C})$ ($s \in S$), obtained from the Kuranishi family, where D is the period domain (see, for detail, [5]).

PROPOSITION (1.17) (*Local Torelli theorem for the restricted family*): *The restriction*

$$\mathrm{res}\, \phi : S^\sigma \to D$$

of the period map ϕ *in* (1.16) *is injective.*

Proof: First of all, note that σ has induced actions on S as above and also on D and that ϕ is σ-equivariant with these induced actions.

Let

$$d\phi(s_0): T_S(s_0) \to T_D(\phi(s_0))$$

be the differential map of the period map ϕ at $s_0 \in S$. Since $T_S(s_0)$ (resp. $T_D(\phi(s_0))$ can be identified with $H^1(X, T_X)$ (resp. $\mathrm{Hom}(P^{2,0}(X), P^{1,1}(X))$), we know, from Theorem (2.14) in [11], that the decomposition of $T_S(s_0)$ and $T_D(\phi(s_0))$ into their eigen spaces under the action of σ are the following:

$$\begin{aligned} &T_S(s_0) = T_1 \oplus T_\omega \oplus T_{\omega^2} \quad \text{with } \dim T_1 = \dim T_\omega = \dim T_{\omega^2} = 6, \\ (1.18) \quad &T_D(\phi(s_0)) = T'_1 \oplus T'_\omega \oplus T'_{\omega^2} \quad \text{with } \dim T'_1 = 8, \\ &\dim T'_\omega = \dim T'_{\omega^2} = 5, \end{aligned}$$

where T_λ (resp. T'_λ) is the λ-eigen subspace of $T_S(s_0)$ (resp. $T_D(\phi(s_0))$).

Since $d\phi(s_0)$ is also σ-equivariant, $d\phi(s_0)$ is compatible with the decompositions in (1.18). Hence, from (1.18), $\mathrm{Ker}\, d\phi(s_0)$ contains at least 2-dimensional subspace of $T_\omega \oplus T_{\omega^2}$. On the other hand, it can be shown easily (cf. [6], [2] or [11]) that $\dim \mathrm{Ker}\, d\phi(s_0) \leqq 2$. Thus, we can conclude that

$$(1.19) \qquad T_1 \cap \mathrm{Ker}\, d\phi(s_0) = \{0\}.$$

Since $T_{S^\sigma}(s_0) = T_1$, (1.19) means that

$$\mathrm{res}\, d\phi(s_0): T_{S^\sigma}(s_0) \to T_D(\phi(s_0))$$

is injective. This shows that

$$\mathrm{res}\, \phi : S^\sigma \to D$$

is injective, because we consider S^σ as germ. Q.E.D.

2. Structure theorem

We continue to use the notation in the previous section.

2.1. Let $X = X_u$ ($u \in U_{15}$). Since $\sigma = (1, \omega, \omega^2, 1, 1)$ (see (1.18)), the fixed points of X by σ satisfy the equations

$$(2.1) \qquad x_0 = y_1 = 0,$$

$$(2.2) \qquad x_0 = y_2 = 0 \quad \text{or}$$

$$(2.3) \qquad y_1 = y_2 = 0.$$

We can calculate easily that

the intersection number of the curves $(x_0 = 0)$ and $(y_i = 0) = 2$ $(i = 1, 2)$
the intersection number of the curves $(y_1 = 0)$ and $(y_2 = 0) = 4$.

Moreover, since $\sigma \in \mathrm{Aut}(X)$ is of finite order, the fixed points locus X^σ of X by σ is smooth. Thus we get that X^σ consists of 8 distinct points. We denote these points by

(2.4) $X = \{D_i, E_i\ (i = 1, 2, 3, 4)\}$, where
D_i $(i = 1, 2)$ satisfy the equations (2.1),
D_i $(i = 3, 4)$ satisfy the equations (2.2) and
E_i $(i = 1, 2, 3, 4)$ satisfy the equations (2.3).

Since we can take x_0z_3/y_2^2, y_1/y_2 (resp. x_0z_3/y_1^2, y_2/y_1; resp. y_1/x_0^2, y_2/x_0^2) as local coordinates of X at D_i $(i = 1, 2)$ (resp. D_i $(i = 3, 4)$ resp. E_i $(i = 1, 2, 3, 4)$), we see that the induced actions of σ on the normal spaces of these points in X are

$$(2.5) \qquad \begin{array}{ll} (\omega^2, \omega^2) & \text{at } D_i\ (i = 1, 2), \\ (\omega, \omega) & \text{at } D_i\ (i = 3, 4) \text{ and} \\ (\omega, \omega^2) & \text{at } E_i\ (i = 1, 2, 3, 4). \end{array}$$

Let

$$(2.6) \qquad \tilde{X} \to X$$

be the blowing-up of X with center X^σ. Denote by

$$(2.7) \qquad \tilde{D}_i \text{ and } \tilde{E}_i \quad (i = 1, 2, 3, 4)$$

the exceptional curves on $\tilde{X}$ corresponding to the points D_i and E_i on X respectively.

The action of σ extends naturally on $\tilde{X}$ so that the morphism (2.6) is σ-equivariant. From (2.5), we see that there are 2 distinct points, say

$$(2.8) \qquad \tilde{E}_{ij} \quad (j = 1, 2),$$

on each $\tilde{E}_i$ which are fixed by σ, and the fixed points locus $\tilde{X}^\sigma$ of $\tilde{X}$ by σ is

$$(2.9) \qquad \tilde{X}^\sigma = \{\tilde{D}_i, \tilde{E}_{ij}\ (i = 1, 2, 3, 4; j = 1, 2)\}.$$

We know, also from (2.5), that the induced action of σ on the normal bundle of each component of $\tilde{X}^\sigma$ in $\tilde{X}$ is

$$(2.10) \qquad \begin{array}{llll} (\omega^2) & \text{along} & \tilde{D}_i & (i = 1, 2), \\ (\omega) & \text{along} & \tilde{D}_i & (i = 3, 4), \\ (\omega, \omega) & \text{at} & \tilde{E}_{i1} & (i = 1, 2, 3, 4) \text{ and} \\ (\omega^2, \omega^2) & \text{at} & \tilde{E}_{i2} & (i = 1, 2, 3, 4). \end{array}$$

Let

$$(2.11) \qquad \hat{X} \to \tilde{X}$$

be the blowing-up of $\tilde{X}$ with center $\tilde{X}^\sigma$. Denote by

$$(2.12) \qquad \hat{D}_i, \hat{E}_i \quad \text{and} \quad \hat{E}_{ij} \quad (i = 1, 2, 3, 4; j = 1, 2)$$

the curves on $\hat{X}$ which are the inverse images of $\tilde{D}_i$, the proper transforms of $\tilde{E}_i$ and the exceptional divisors corresponding to $\tilde{E}_{ij}$ respectively.

The action of σ extends again to $\hat{X}$ and we see, from (2.10), that the fixed points locus $\hat{X}^\sigma$ of $\hat{X}$ by σ is now a disjoint union of 12 curves, i.e.

$$(2.13) \qquad \hat{X}^\sigma = \{\hat{D}_i, \hat{E}_{ij}\ (i = 1, 2, 3, 4; j = 1, 2)\}.$$

From (2.10) again, we know that the induced action of σ on the normal bundle of each component of $\hat{X}^\sigma$ in $\hat{X}$ is the following:

$$(2.14) \qquad \begin{array}{l} (\omega) \text{ along } \hat{D}_i\ (i = 3, 4) \text{ and along } \hat{E}_{i1}\ (i = 1, 2, 3, 4). \\ (\omega^2) \text{ along } \hat{D}_i\ (i = 1, 2) \text{ and along } \hat{E}_{i2}\ (i = 1, 2, 3, 4). \end{array}$$

We denote by

$$(2.15) \qquad p : \hat{X} \to X$$

the composite morphism of (2.11) and (2.6). Note that p is σ-equivariant.

We can calculate easily the self-intersection numbers of the exceptional curves on $\hat{X}$ of the morphism p:

$$(2.16)\quad (\hat{D}_i)^2 = (\hat{E}_{ij})^2 = -1, \quad (\hat{E}_i)^2 = -3 \quad (i = 1, 2, 3, 4; j = 1, 2).$$

Denote by

$$(2.17)\qquad C \text{ and } \hat{C}$$

the canonical divisor of X and its proper transform by p in (2.15). Since $x_0 = 0$ is the homogeneous equation of C in X, C contains 4 points D_i $(i = 1, 2, 3, 4)$ in (2.4). From this fact we get that

$$(2.18)\qquad (\hat{C})^2 = -3.$$

2.2. Since $\sigma \in \mathrm{Aut}(\hat{X})$ is of order 3 and $\hat{X}^\sigma$ is of pure codimension 1, we get a ramified triple covering

$$(2.19)\qquad r: \hat{X} \to \hat{Y},$$

where $\hat{Y} = \hat{X}/\langle\sigma\rangle$ is smooth. We denote by $\hat{R}$ the ramification locus and by $\hat{B}$ the branch locus of r, i.e.

$$(2.20)\qquad \hat{R} = \hat{X}^\sigma = \sum_{1 \leq i \leq 4} \hat{D}_i + \sum_{1 \leq i \leq 4, j=1,2} \hat{E}_{ij} \quad \text{and} \quad \hat{B} = r(\hat{R}).$$

We consider $\hat{R}$ and $\hat{B}$ as reduced curves.

We use the notation

$$(2.21)\quad \hat{C}' = r(\hat{C}), \quad \hat{D}'_i = r(\hat{D}_i), \quad \hat{E}'_i = r(\hat{E}_i) \quad \text{and} \quad \hat{E}'_{ij} = r(\hat{E}_{ij}),$$

where all these curves are considered as reduced curves on $\hat{Y}$.

LEMMA (2.22): *All the curves in* (2.21) *are smooth, irreducible, rational curves with self-intersection numbers*

$$(\hat{C}')^2 = (\hat{E}'_i)^2 = -1 \quad \text{and} \quad (\hat{D}'_i)^2 = (\hat{E}'_{ij})^2 = -3 \quad (i = 1, 2, 3, 4; j = 1, 2).$$

PROOF: We see easily that C is a smooth curve of genus 2 by the Jacobian criterion and adjunction formula. Hence, so is $\hat{C}$, because $\hat{C}$ is isomorphic to C. From the construction, we know that

$$\hat{C} \to \hat{C}'$$

is a triple covering ramified at 4 distinct points $\hat{C} \cap (\Sigma_{1 \leqq i \leqq 4} \hat{D}_i)$. Hence, we see that $\hat{C}'$ is a smooth, irreducible, rational curve by the Hurwitz formula.

In the same way, by using the fact that

$$\hat{E}_i \to \hat{E}'_i$$

is a triple covering ramified at 2 distinct points $\hat{E}_i \cap (\hat{E}_{i1} + \hat{E}_{i2})$, we can prove that $\hat{E}'_i$ are also smooth, irreducible, rational curves.

The same assertion for the curves $\hat{D}'_i$ and $\hat{E}'_{ij}$ is trivial because they are isomorphic to $\hat{D}_i$ and $\hat{E}_{ij}$ respectively.

As for the statement for the self-intersection numbers, we can obtain immediately from (2.16) and (2.18) by the projection formula. Q.E.D.

2.3. Let

$$q : \hat{Y} \to Y \tag{2.23}$$

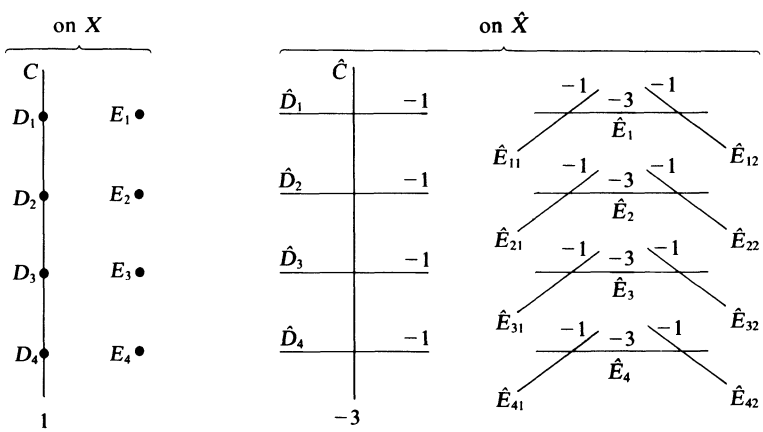
be the morphism obtained by blowing-down the exceptional curves of the first kind $\hat{C}'$ and $\hat{E}'_i$ $(i = 1, 2, 3, 4)$. Set

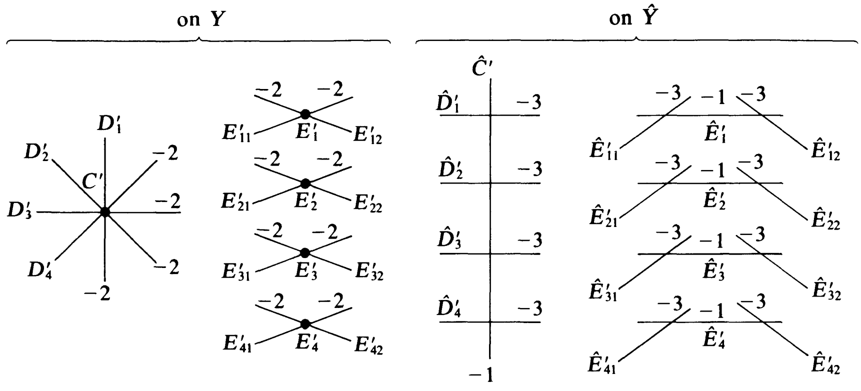
$$C' = q(\hat{C}'), \quad E'_i = q(\hat{E}'_i), \quad D'_i = q(\hat{D}'_i) \quad \text{and} \quad E'_{ij} = q(\hat{E}'_{ij}) \qquad (i = 1, 2, 3, 4; j = 1, 2). \tag{2.24}$$

Then, C' and E'_i are points, and D'_i and E'_{ij} are smooth, irreducible, rational curves with self-intersection number -2.

We write down the configurations of the points and the curves appeared in 2.1, 2.2 and 2.3 with their self-intersection numbers:

(2.25)





2.4. Now we can state the relation of our surfaces with K3 surfaces. We use the notation in 2.1, 2.2 and 2.3.

PROPOSITION (2.26) (Structure theorem): *Set $X = X_u$ $(u \in U^\sigma)$. Then, starting from X, we can construct a diagram*

$$\begin{array}{ccc} X & \xleftarrow{\ p\ } & \hat{X} \\ & & \big\downarrow r \\ Y & \xleftarrow{\ q\ } & \hat{Y} \end{array}$$

where

(i) *p is the morphism in* (2.15), *i.e. the morphism obtained by a sequence of blowings-up at the fixed points by σ, so that the fixed points locus in $\hat{X}$ under the induced action of σ is of pure codimension* 1,

(ii) *r is the morphism in* (2.19), *i.e. the natural projection onto the quotient of $\hat{X}$ by the group $\langle\sigma\rangle$ generated by σ, and*

(iii) *q is the morphism in* (2.23), *i.e. the morphism obtained by blowing-down onto the minimal model Y.*

Moreover, we have that

(iv) *Y is a minimal K3 surface,*

(v) $3(\Sigma_{1\le i\le 4} D_i') - 2(\Sigma_{1\le i\le 4, j=1,2} E_{ij}')$ *is an ample divisor on Y, and*

(vi) $\pi_1(\hat{X} - \hat{R}) = \{1\}$, *where $\hat{R}$ is the ramification locus of r.*

PROOF: The remaining things to prove are the assertions (iv), (v) and (vi).

First, we will prove (iv). By the construction of Y, it is clear that the unique holomorphic 2-form on X, vanishing on C and σ-invariant, gives a nowhere vanishing holomorphic 2-form on Y. Combining this with $q(Y) \leq q(X) = 0$, we get (iv).

For the proof of (v), we use the configuration (2.25). First of all, we see that

$$(2.27) \qquad \left(3\left(\sum_{1\leqq i\leqq 4} D'_i\right) - 2\left(\sum_{1\leqq i\leqq 4,\, j=1,2} E'_{ij}\right)\right)^2 = 9\left(\sum D'_i\right)^2 + 4\left(\sum E'_{ij}\right)^2 = 4 > 0.$$

By the assumption, C is ample and hence so is

$$p^*(4C) - \left(\sum \hat{D}_i + \sum \hat{E}_i + 2\left(\sum \hat{E}_{ij}\right)\right) = 4\hat{C} - \left(\sum \hat{E}_i\right) + 3\left(\sum \hat{D}_i\right) - 2\left(\sum \hat{E}_{ij}\right).$$

Since r is a finite morphism and

$$3\left(4\hat{C} - \left(\sum \hat{E}_i\right) + 3\left(\sum \hat{D}_i\right) - 2\left(\sum \hat{E}_{ij}\right)\right) = r^*\left(12\hat{C}' - 3\left(\sum \hat{E}'_i\right) + 3\left(\sum \hat{D}'_i\right) - 2\left(\sum \hat{E}'_{ij}\right)\right),$$

we see that

$$12\hat{C}' - 3\left(\sum \hat{E}'_i\right) + 3\left(\sum \hat{D}'_i\right) - 2\left(\sum \hat{E}'_{ij}\right)$$

is an ample divisor on $\hat{Y}$. Denote this divisor by F. Since $\hat{C}'$ and $\hat{E}'_i$ are the exceptional curves of the morphism q, we see, by the Nakai criterion of ampleness for F, that for any integral curve Z on Y

$$(2.28) \qquad \left(3\left(\sum D'_i\right) - 2\left(\sum E'_{ij}\right), Z\right) = \left(q^*\left(3\left(\sum D'_i\right) - 2\left(\sum E'_{ij}\right)\right), q^*Z\right) = (F, q^*Z) > 0.$$

Thus, the assertion (v) follows from (2.27) and (2.28) by the Nakai criterion again.

Finally, we will prove (vi). We use the result in [2]:

$$\pi_1(X) = \{1\}.$$

Since X^σ consists of finite points, we see that

$$\pi_1(X - X^\sigma) = \pi_1(X) = \{1\}. \tag{2.29}$$

By using (2.29) and the following diagram

$$\begin{array}{c} X - X^\sigma \xleftarrow{\sim} \hat{X} - \left(\hat{R} + \sum_{1 \leqq i \leqq 4} \hat{E}_i\right) \\ \cap \\ \hat{X} - \hat{R}, \end{array}$$

we get our assertion (vi). Q.E.D.

3. Torelli theorem

In this section, we will prove the Torelli theorem for the surfaces with $p_g = c_1^2 = 1$, with an ample canonical divisor and with an automorphism of order 3 acting trivially on the holomorphic 2-forms.

We continue to use the notation in the previous sections.

First, we give an elementary lemma which can be verified easily by a standard argument using the discreteness of integral homology groups.

LEMMA (3.1): *Let ψ be a morphism of smooth families $\{V_t\}_{t\in T}$ and $\{W_t\}_{t\in T}$ of compact, complex manifolds over a complex manifold T and suppose we are given a path α in T joining two points t and t' in T.*

Then, we have a commutative diagram

$$\begin{array}{ccc} H_n(V_t, \mathbf{Z}) & \xrightarrow{\psi_{t*}} & H_n(W_t, \mathbf{Z}) \\ {\scriptstyle \alpha_*}\downarrow \wr & & {\scriptstyle \alpha_*}\downarrow \wr \\ H_n(V_{t'}, \mathbf{Z}) & \xrightarrow{\psi_{t'*}} & H_n(W_{t'}, \mathbf{Z}) \end{array}$$

for all n, where α_* is the isomorphism obtained by a C^∞-trivialization along the path α, and this α_* is compatible with intersection products.

Let $\pi'_{15}: \mathscr{X}'_{15} \to U_{15}$ be the family in (1.10). For any two points $u_k \in U_{15}$ $(k = 1, 2)$, taking a path $\tilde{\tau}$ in U_{15} joining u_1 and u_2 and applying Lemma (3.1), we get a commutative diagram

(3.2)
$$\begin{array}{ccc} H_2(X_1, \mathbf{Z}) & \xrightarrow{1-\sigma} & H_2(X_1, \mathbf{Z}) \\ {\scriptstyle \tau_*}\downarrow\wr & & {\scriptstyle \tau_*}\downarrow\wr \\ H_2(X_2, \mathbf{Z}) & \xrightarrow{1-\sigma} & H_2(X_2, \mathbf{Z}) \end{array}$$

where $X_k = \pi'^{-1}_{15}(u_k)$ and τ_* is the isometry obtained from the path $\tilde{\tau}$. Hence, we get the induced isometry

(3.3)
$$\tau_*: H_2(X_1, \mathbf{Z})^\sigma \xrightarrow{\sim} H_2(X_2, \mathbf{Z})^\sigma$$

of the kernels of $1 - \sigma$ in (3.2).

THEOREM (3.4): *Suppose we are given two points $u_k \in U_{15}$ $(k = 1, 2)$ and a path $\tilde{\tau}$ in U_{15} joining u_1 and u_2, and suppose the induced isometry τ_* in* (3.3) *preserves the periods of integrals of the holomolphic 2-forms ω_{X_k} on $X_k = \pi'^{-1}_{15}(u_k)$ $(k = 1, 2)$, i.e.*

$$\int_{\tau_*\gamma} \omega_{X_2} = (constant) \int_\gamma \omega_{X_1}$$

for all $\gamma \in H_2(X_1, \mathbf{Z})^\sigma$, where (constant) is independent of γ.

Then, there exists an isomorphism

$$\tau: X_1 \xrightarrow{\sim} X_2$$

inducing the given τ_* and such τ is uniquely determined up to composition with an element of the group $\langle\sigma\rangle$ generated by σ. We have also $\tau\sigma\tau^{-1} = \sigma$ or σ^2.

PROOF: Starting from the family (1.10), we can construct, in a similar way as in the section 2, a commutative diagram

(3.5)
$$\begin{tikzcd} \mathscr{X}'_{15} \arrow[drr, "\pi'_{15}"'] & \hat{\mathscr{X}} \arrow[l, "\hat{p}"'] \arrow[r, "\hat{r}"] \arrow[dr, "\hat{\pi}"] & \hat{\mathscr{Y}} \arrow[r, "\hat{q}"] \arrow[d, "\hat{\pi}'"] & \mathscr{Y} \arrow[dl, "\pi'"] \\ & & U_{15} & \end{tikzcd}$$

whose fibre over every point of U_{15} satisfies the properties (i) to (vi) in Proposition (2.26). In fact, $\tilde{p}$ and $\tilde{r}$ in (3.5) can be constructed just in the same way as p and r in the section 2, and the construction of $\tilde{q}$ in (3.5) is justified by the result in [3].

For $k = 1, 2$, set $\hat{X}_k = \hat{\pi}^{-1}(u_k)$, $\hat{Y}_k = \hat{\pi}'^{-1}(u_k)$, *and* $Y_k = \pi'^{-1}(u_k)$, and let $p_k : \hat{X}_k \to X_k$, $r_k : \hat{X}_k \to \hat{Y}_k$ and $q_k : \hat{Y}_k \to Y_k$ be the restrictions to the fibres of the morphisms $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ in (3.5) respectively. We denote by $\hat{D}_i^{(k)}$, $\hat{E}_i^{(k)}$ and $\hat{E}_{ij}^{(k)}$ the corresponding curves on $\hat{X}_k$ and by $C'^{(k)}$, $D_i'^{(k)}$, $E_i'^{(k)}$ and $E_{ij}'^{(k)}$ the corresponding points and curves on Y_k $(k = 1, 2)$ constructed in the section 2. Denote also by $\hat{R}_k$ and $\hat{B}_k$ the ramification locus and the branch locus of the triple covering $r_k : \hat{X}_k \to \hat{Y}_k$ $(k = 1, 2)$. For a divisor F on a surface, we denote by $[F]$ the integral homology class represented by F.

Then, by Lemma (3.1), we get, from (3.5), the commutative diagram of homology groups:

$$\begin{array}{ccccccc}
H_2(X_1, \mathbf{Z})^\sigma & \xleftarrow{p_{1*}} & H_2(\hat{X}_1, \mathbf{Z})^\sigma & \xrightarrow{r_{1*}} & H_2(\hat{Y}_1, \mathbf{Z}) & \xrightarrow{q_{1*}} & H_2(Y_1, \mathbf{Z}) \\
\tau_* \downarrow \wr & & \hat{\tau}_* \downarrow \wr & & \hat{\tau}'_* \downarrow \wr & & \tau'_* \downarrow \wr \\
H_2(X_2, \mathbf{Z})^\sigma & \xleftarrow{p_{2*}} & H_2(\hat{X}_2, \mathbf{Z})^\sigma & \xrightarrow{r_{2*}} & H_2(\hat{Y}_2, \mathbf{Z}) & \xrightarrow{q_{2*}} & H_2(Y_2, \mathbf{Z})
\end{array}$$

(3.6)

$$\begin{array}{ccccccc}
H_2(X_1, \mathbf{Z})^\sigma & \xleftarrow{p_1^*} & H_2(\hat{X}_1, \mathbf{Z})^\sigma & \xrightarrow{r_1^*} & H_2(\hat{Y}_1, \mathbf{Z}) & \xrightarrow{q_1^*} & H_2(Y_1, \mathbf{Z}) \\
\tau_* \downarrow \wr & & \hat{\tau}_* \downarrow \wr & & \hat{\tau}'_* \downarrow \wr & & \tau'_* \downarrow \wr \\
H_2(X_2, \mathbf{Z})^\sigma & \xleftarrow{p_2^*} & H_2(\hat{X}_2, \mathbf{Z})^\sigma & \xrightarrow{r_2^*} & H_2(\hat{Y}_2, \mathbf{Z}) & \xrightarrow{q_2^*} & H_2(Y_2, \mathbf{Z})
\end{array}$$

where $\hat{\tau}_*$, $\hat{\tau}'_*$ and τ'_* are the induced isometries, like τ_*, from the path $\tilde{\tau}$. By our construction of (3.5), we see that

$$(3.7) \quad \begin{aligned} &\hat{\tau}_*([\hat{D}_i^{(1)}]) = [\hat{D}_i^{(2)}], \quad \hat{\tau}_*([\hat{E}_i^{(1)}]) = [\hat{E}_i^{(2)}]), \quad \hat{\tau}_*([\hat{E}_{ij}^{(1)}]) = [\hat{E}_{ij}^{(2)}], \\ &\hat{\tau}'_*([\hat{B}_1]) = [\hat{B}_2], \quad \tau'_*([D_i'^{(1)}]) = [D_i'^{(2)}], \quad \tau'_*([E_{ij}'^{(1)}]) = [E_{ij}'^{(2)}]. \end{aligned}$$

Note also that $p_{k*}p_k^* = id$, $q_{k*}q_k^* = id$, $r_{k*}r_k^* = 3id$ and $r_k^* r_{k*} = 3id$ $(k = 1, 2)$.

Let $\omega_{\hat{X}_k}$ (resp. $\omega_{\hat{Y}_k}$, ω_{Y_k}) be the holomorphic 2-form on $\hat{X}_k$ (resp.

$\hat{Y}_k$, Y_k) induced from ω_{X_k} $(k = 1, 2)$. Since

$$\int_{\gamma} \omega_{Y_k} = \int_{q_k^* \gamma} \omega_{\hat{Y}_k} = 3 \int_{r_k^* q_k^* \gamma} \omega_{\hat{X}_k} = 3 \int_{p_{k*} r_k^* q_k^* \gamma} \omega_{X_k}$$

for any $\gamma \in H_2(Y_k, \mathbf{Z})$, we can deduce, by (3.6), the property

$$\int_{\tau'_* \gamma} \omega_{Y_2} = (\text{constant}) \int_{\gamma} \omega_{Y_1} \quad \text{for all } \gamma \in H_2(Y_1, \mathbf{Z})$$

from that on X_k.

Since

$$\tau'_* \left(\left[3 \left\{ \sum_i D_i'^{(1)} \right) - 2 \left(\sum_{i,j} E_{ij}'^{(1)} \right) \right] \right) = \left[3 \left(\sum_i E_i'^{(2)} \right) - 2 \left(\sum_{i,j} E_{ij}'^{(2)} \right) \right]$$

from (3.7), we see, by (v) in Proposition (2.26), that τ'_* sends some ample divisor class on Y_1 to an ample divisor class on Y_2.

Hence, we can apply the Strong Torelli Theorem for algebraic K3 surfaces proved and supplemented in [8], [1] and [7] to our case, and we see that there exists uniquely the isomorphism

$$\tau' : Y_1 \xrightarrow{\sim} Y_2$$

inducing the isometry τ'_* in (3.6).

Considering (3.7) and intersection numbers, we can observe easily

$$\tau'(D_i'^{(1)}) = D_i'^{(2)} \quad \text{and} \quad \tau'(E_{ij}'^{(1)}) = E_{ij}'^{(2)}$$

and hence, in particular,

$$\tau'(C'^{(1)}) = C'^{(2)} \quad \text{and} \quad \tau'(E_i'^{(1)}) = E_i'^{(2)}.$$

Therefore, by the construction of $q_k : \hat{Y}_k \to Y_k$, τ' can be lifted uniquely to an isomorphism

$$\hat{\tau}' : \hat{Y}_1 \xrightarrow{\sim} \hat{Y}_2$$

inducing the isometry $\hat{\tau}'_*$ in (3.6).

Considering (3.7) and intersection numbers again, we see

$$\hat{\tau}'(\hat{B}_1) = \hat{B}_2.$$

Since we know that $r_k : \hat{X}_k - \hat{R}_k \to \hat{Y}_k - \hat{B}_k$ are universal coverings by (vi) in Proposition (2.26), there exists an isomorphism

$$\hat{\tau} : \hat{X}_1 - \hat{R}_1 \xrightarrow{\sim} \hat{X}_2 - \hat{R}_2$$

compatible with $\hat{\tau}'$. Such $\hat{\tau}$ are unique up to the covering transformation group $\langle \sigma \rangle$. Now, by the Riemann Extension Theorem, $\hat{\tau}$ extends uniquely to an isomorphism

$$\hat{\tau} : \hat{X}_1 \xrightarrow{\sim} \hat{X}_2,$$

where we abuse the notation $\hat{\tau}$. $\hat{\tau}$ is compatible with $\hat{\tau}'$ and hence induces the isometry $\hat{\tau}_*$ in (3.6).

By the argument on intersection numbers, we get, from (3.7), that

$$\hat{\tau}(\hat{D}_i^{(1)}) = \hat{D}_i^{(2)}, \quad \hat{\tau}(\hat{E}_{ij}^{(1)}) = \hat{E}_{ij}^{(12)} \quad \text{and} \quad \hat{\tau}(\hat{E}_i^{(1)}) = \hat{E}_i^{(2)}.$$

Hence, $\hat{\tau}$ descends uniquely to an isomorphism

$$\tau : X_1 \xrightarrow{\sim} X_2$$

inducing the given isometry τ_*.

The other assertion follows easily. Q.E.D.

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